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# Ergodicity of the Finite Dimensional Approximation of the 3D Navier–Stokes Equations Forced by a Degenerate Noise

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We prove ergodicity of the finite dimensional approximations of the three dimensional Navier–Stokes equations, driven by a random force. The forcing noise acts only on a few modes and some algebraic conditions on the forced modes are found that imply the ergodicity. The convergence rate to the unique invariant measure is shown to be exponential.

**KEY WORDS:** Navier–Stokes equations; invariant measure; ergodicity; Hörmander condition; controllability; Lie saturates; Lyapunov function.

## 1. INTRODUCTION

The uniqueness of statistical steady states for the Navier–Stokes equations is an important problem in the mathematical theory of turbulence. The question is completely open in dimension three, due to the lack of uniqueness of the equations. In fact, there is no way yet to give meaning to the mathematical objects involved in the subject.

In the present paper the property of ergodicity is proved for the finite dimensional approximations of the three-dimensional Navier–Stokes equations, driven by a random force. The same problem has been solved in two dimensions by E and Mattingly.[(3)]

Such a result can have a qualitative interest for the statistical behaviour of an incompressible fluid. Indeed, if the Kolmogorov theory of turbulence is taken into account, one can believe that the cascade of energy, responsible of the transport of the energy through the scales, is effective in

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the inertial range. At smaller scales, only the dissipation ends up to be relevant. Hence the long-time statistical properties of the fluid can be sufficiently depicted by the low modes of the velocity field. In some sense, if the ultraviolet cut-off is sufficiently large, in order to capture all the important modes, the corresponding invariant measure gives the real behaviour of the fluid. In view of these considerations, the conclusions of the paper can give both a hint and a possible starting point for the analysis of the infinite dimensional case (the corresponding analysis, in the deterministic setting, turns out to be true, see, for example, Constantin, Foias, and Temam[1]).

We consider a finite dimensional truncation of the three dimensional Navier–Stokes equations, driven by a random force, with periodic boundary conditions. The method of the proof of ergodicity is classical and it consists of two steps. First we prove that the transition probability densities are regular, by checking that the diffusion operator is hypoelliptic (the Hörmander condition). Then we show that the Markov process is irreducible, in the sense that each open set is visited with positive probability at each time. To this end, we study the associated control problem (see Section 6) with the help of geometric control theory.

A general result of Jurdjevic and Kupka[10] for polynomial control systems with leading terms of odd degree says that hypoellipticity is equivalent to controllability. The claim is not true in general, when dealing with polynomials of even degree, like the one considered here. Roughly speaking, the positive terms of a even degree polynomial non-linearity impose on the process some privileged direction to be followed, and consequently some unavailable directions. The main point in this paper is that the geometrical properties of the Navier–Stokes non-linearity can be used to show that the *obstructions* induced by the positive terms do not prevent the process from visiting any open set of the whole state space with positive probability.

The controllability property ensures in particular that the invariant measure is supported on the whole state space. Notice that irreducibility for the infinite dimensional equations was originally proved by Flandoli,[5] but under the assumption that the noise acts on all modes.

Both these properties, strong Feller and irreducibility, are implied by an algebraic condition on the set of indices corresponding to the modes forced by the noise. The condition *essentially* means that it is possible to obtain any index as a sum of some of the forced indices. One can see this mechanism as a geometrical realisation of the cascade of energy, since the non-linear term transmits the random forcing from the few forced modes to all the other modes. As an example we show that the algebraic condition is satisfied if the three lowest modes are forced.

Recently, many authors have applied the techniques we have used here, such us the hypoellipticity for degenerate diffusions, or the general theory for Markov chains and Markov processes collected and developed by Meyn and Tweedie (see, e.g., their book[11]). Among many others we quote the papers by E and Mattingly,[3] Eckmann and Hairer,[4] Hairer,[7] Rey-Bellet and Thomas,[14] and some of the references therein.

The paper is organised as follows. In the first section the main definitions are given, together with the statements of the main results and an outline of their proofs. The technical computations and the precise statement of some hypotheses are postponed in the following sections. The aim is to give a light presentation of the main ideas, without all the technicalities, which are then reserved to the interested readers.

## 2. THE MAIN THEOREM

We consider the stochastic Navier–Stokes equations with additive noise

$$\begin{aligned} du &= (\nu \Delta u - (u \cdot \nabla)\, u - \nabla P)\, dt + dB_t \\ \operatorname{div} u &= 0, \end{aligned}$$

in the domain $[0, 2\pi]^3$, with periodic boundary conditions, where $u$ is the velocity field and $P$ is the pressure field, and $B_t$ is a Brownian motion. As usual, the equations are projected on the space of divergence-free vector fields, in order to cause the pressure to disappear from the equations. If we write the equations in the Fourier components, we obtain the following infinite system of stochastic differential equations

$$du_{\mathbf{k}} = \left[ -\nu\, |\mathbf{k}|^2\, u_{\mathbf{k}} - \mathrm{i} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k} \cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\, \mathbf{k} \right) \right] dt + q_{\mathbf{k}}\, d\beta_t^{\mathbf{k}},$$

for $\mathbf{k} \in \mathbf{Z}^3$, with the constraint $u_{\mathbf{k}} \cdot \mathbf{k} = 0$ (it comes from the divergence-free condition). We have made some simplifying assumptions on the noise: we assume that the noise takes values in the space of divergence-free vector fields and that the covariance is diagonal in the Fourier components (the assumptions will be stated more clearly in Section 3.1).

In order to state the problem of the finite dimensional approximation, fix a threshold $N$ and consider the finite subset of indices

$$\mathscr{K}_N = \{ \mathbf{k} \in \mathbf{Z}^3 \mid |\mathbf{k}| \leqslant N,\ \mathbf{k} \neq (0, 0, 0) \}.$$

The finite dimensional system obtained is the following

$$du_{\mathbf{k}} = \Big[ -\nu\, |\mathbf{k}|^2\, u_{\mathbf{k}} - \mathrm{i} \sum_{\substack{\mathbf{h},\,\mathbf{l} \in \mathscr{K}_N \\ \mathbf{h}+\mathbf{l}=\mathbf{k}}} (\mathbf{k}\cdot u_{\mathbf{h}}) \Big( u_{\mathbf{l}} - \frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\,\mathbf{k} \Big) \Big]\, dt + q_{\mathbf{k}}\, d\beta^{\mathbf{k}}_t, \quad (2.1)$$

with $\mathbf{k} \in \mathscr{K}_N$ (a formal derivation is given in Section 3). We will use the real variables $r_{\mathbf{k}},\, s_{\mathbf{k}} \in \mathbf{R}^3$, where $u_{\mathbf{k}} = r_{\mathbf{k}} + \mathrm{i} s_{\mathbf{k}}$, rather than the complex variables $u_{\mathbf{k}}$, so that the equations are briefly written as

$$\begin{cases} dr^i_{\mathbf{k}} = F_{r^i_{\mathbf{k}}}(r, s)\, dt + q^r_{\mathbf{k}}\, d\beta^{\mathbf{k}}_t, \\ ds^i_{\mathbf{k}} = F_{s^i_{\mathbf{k}}}(r, s)\, dt + q^s_{\mathbf{k}}\, d\beta^{\mathbf{k}}_t, \end{cases} \qquad \mathbf{k} \in \mathscr{K}_N, \quad i = 1, 2, 3$$

and $q_{\mathbf{k}} = q^r_{\mathbf{k}} + \mathrm{i} q^s_{\mathbf{k}}$. Since $u_{-\mathbf{k}} = \overline{u_{\mathbf{k}}}$, the set of indices $\mathscr{K}_N$ is redundant, hence we take a smaller set $\widetilde{\mathscr{K}}$, which takes into account the symmetries.

The solution $(r(t), s(t))$ of the above stochastic equations is a Markov process on the state space

$$U = \bigoplus_{\mathbf{k} \in \widetilde{\mathscr{K}}} (R_{\mathbf{k}} \oplus S_{\mathbf{k}}),$$

where $R_{\mathbf{k}}$ and $S_{\mathbf{k}}$ enclose the divergence-free condition $r_{\mathbf{k}} \cdot \mathbf{k} = s_{\mathbf{k}} \cdot \mathbf{k} = 0$ (see also (4.1) and the following formulas). We denote by $P_t$ the transition semigroup

$$P_t \varphi(r_0, s_0) = \mathbf{E}_{(r_0, s_0)}[\varphi(r(t), s(t))]$$

with generator

$$\mathscr{L} = F_0 + \tfrac{1}{2} \sum_{\substack{\mathbf{k} \in \widetilde{\mathscr{K}} \\ i = 1, 2, 3}} ({X^r_{\mathbf{k}, i}}^2 + {X^s_{\mathbf{k}, i}}^2) \quad (2.2)$$

where

$$F_0 = \sum_{\mathbf{k} \in \widetilde{\mathscr{K}}} \sum_{i=1}^{3} F_{r^i_{\mathbf{k}}} \frac{\partial}{\partial r^i_{\mathbf{k}}} + F_{s^i_{\mathbf{k}}} \frac{\partial}{\partial s^i_{\mathbf{k}}}, \quad (2.3)$$

and

$$X^r_{\mathbf{k}, i} = \sum_{j=1}^{3} q^r_{\mathbf{k}, ij} \frac{\partial}{\partial r^j_{\mathbf{k}}}, \qquad X^s_{\mathbf{k}, i} = \sum_{j=1}^{3} q^s_{\mathbf{k}, ij} \frac{\partial}{\partial s^j_{\mathbf{k}}}, \quad (2.4)$$

and by $P_t((r, s), \cdot)$ the transition probability.

The main assumption we take on the noise is that it acts on a small set of modes. We consider the set $\mathscr{N}$ of indices whose corresponding Fourier components are forced by the noise. We assume that $\mathcal{N}$ is a *determining set of indices*, as defined in Section 5, which *essentially* means that each index in $\mathcal{K}_N$ can be obtained as the sum of elements of $\mathcal{N}$. In other words $\mathcal{N}$ should be an algebraic system of generators of $\mathbf{Z}^3$. In Section 5 we will give some heuristic justifications to such claim. As a *working example*, Proposition 5.3 shows that any set $\mathcal{N}$ containing the three indices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is a determining set of indices.

Here we are interested in stating the main result of the paper, namely the ergodicity of the finite dimensional approximation (2.1)

**Theorem 2.1.** Assume that the Brownian motion $B_t$ satisfies the assumptions in Section 3.1 and that the set $\mathcal{N}$ defined above is a determining set of indices. Then the system (2.1) admits a unique invariant measure.

Moreover, the unique invariant measure is supported on the whole state space or, in other words, it gives positive mass to each open set.

*Proof.* First, we prove the existence of the invariant measure. The method is classical and based on the Krylov–Bogoliubov method (see, for example, Theorem 3.1.1 of Da Prato and Zabczyk[2]). The compactness follows by the following argument. Let $\|u\|^2 = \sum_{\mathbf{k} \in \tilde{\mathcal{K}}} |u_{\mathbf{k}}|^2$, then by Itô formula (using also the first property of Lemma 7.1),

$$\begin{aligned} d\,\|u(t)\|^2 &= \sum_{\mathbf{k} \in \tilde{\mathcal{K}}} (2\overline{u_{\mathbf{k}}} \cdot F_{\mathbf{k}}(u) + \mathrm{Tr}(q_{\mathbf{k}}^T \cdot \overline{q_{\mathbf{k}}}))\,dt + 2 \sum_{\mathbf{k} \in \tilde{\mathcal{K}}} \overline{u_{\mathbf{k}}} \cdot q_{\mathbf{k}}\,d\beta_t^{\mathbf{k}} \\ &= -2\nu \sum_{\mathbf{k} \in \tilde{\mathcal{K}}} |\mathbf{k}|^2\,|u_{\mathbf{k}}|^2\,dt + 2 \sum_{\mathbf{k} \in \tilde{\mathcal{K}}} \overline{u_{\mathbf{k}}} \cdot q_{\mathbf{k}}\,d\beta_t^{\mathbf{k}} + \sigma^2\,dt, \end{aligned}$$

where $\sigma^2$ is the variance of the Brownian motion $B_t$, so that

$$\mathbf{E}\,\|u(t)\|^2 + 2\nu \int_0^t \|u(s)\|^2\,ds \leqslant \mathbf{E}\,\|u(0)\| + \sigma^2 t$$

and by Gronwall lemma $\mathbf{E}\,\|u(t)\| \leqslant \mathbf{E}\,\|u(0)\| + \frac{\sigma^2}{2\nu}$.

Uniqueness of the invariant measure is proved by means of the Doob uniqueness theorem (see, for example, Theorem 4.2.1 of ref. 2). We just need to show that the transition semigroup generated by the dynamics (2.1) is strongly Feller and irreducible.

A Markov semigroup $P_t$ is strongly Feller if $P_t\varphi$ is bounded continuous in time and space when $\varphi$ is bounded measurable. By a theorem of Stroock,[15] the transition semigroup is strongly Feller if the *Hörmander condition* holds: the Lie algebra generated by the vector fields in (2.3) and (2.4), evaluated at each point, is the state space $U$. Since $\mathcal{N}$ is a determining set of indices, from Lemma 4.2 it follows that the constant vector fields of the generated Lie algebra span $U$.

A Markov semigroup is irreducible if it gives positive mass to any open set for each initial condition and each time. It is well known (see Stroock and Varadhan[16]) that irreducibility is true if the control problem (see Eqs. (6.1)) associated to problem (2.1) is controllable. The last statement follows from Theorem 6.5.

Finally, the irreducibility property implies also that the support of the invariant measure is the whole state space. ∎

The next theorem shows, by means of general techniques developed in Meyn and Tweedie,[12, 13] that the finite approximation of Navier–Stokes equations has good dissipation properties, strong enough to ensure the exponential mixing of the dynamics given by the Markov process. In order to state the result, define, for any measurable function $f \geqslant 1$ and any signed measure $\mu$ on the Borel sets of $U$,

$$\|\mu\|_f = \sup_{|g| \leqslant f} \left| \int g(x)\, \mu(dx) \right|,$$

and set

$$V(r, s) = \sum_{\mathbf{k} \in \widetilde{\mathcal{K}}} \sum_{i=1,2,3} (r_{\mathbf{k}}^{i^2} + s_{\mathbf{k}}^{i^2}), \qquad (r, s) \in U.$$

**Theorem 2.2.** Under the assumptions of the previous theorem, let $\pi$ be the unique invariant measure. Then there are positive constants $C$ and $\rho$ such that for each initial condition $(r_0, s_0) \in U$,

$$\|P_t((r_0, s_0), \cdot) - \pi\|_f \leqslant C \mathrm{e}^{-\rho t} \left( 1 + V(r_0, s_0) + \frac{\sigma^2}{2\nu} \right), \qquad t > 0,$$

where $f = 1 + V$.

## 3. THE NAVIER-STOKES EQUATIONS IN THE FOURIER COORDINATES

In this section we derive the equations of the finite dimensional approximations of the stochastic Navier–Stokes equations, with additive noise,

$$du = (\nu \Delta u - (u \cdot \nabla)\, u - \nabla P)\, dt + dB_t$$

$$\operatorname{div} u = 0,$$

in the domain $[0, 2\pi]^3$, with periodic boundary conditions, in the Fourier components.

Consider the Fourier basis $(\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x})_{\mathbf{k}\in\mathbf{Z}^3}$ of $L^2([0, 2\pi]^3)$. First, assume that the applied random force has zero average, so that the centre of mass of the fluid moves with constant velocity. Hence, without loss of generality, we can assume that

$$u_0 = P_0 = 0.$$

The projection onto the space of divergence-free vector fields is defined as

$$\mathscr{P}(a\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}) = \left(a - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\cdot a\right)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} = \left(a - \frac{a\cdot\mathbf{k}}{|\mathbf{k}|^2}\mathbf{k}\right)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x},$$

where $|\cdot|$ is the Euclidean norm in $\mathbf{R}^3$. Notice that

$$\operatorname{div} u = 0 \qquad \text{means} \quad \mathbf{k}\cdot u_{\mathbf{k}} = 0 \qquad \text{for each } \mathbf{k}.$$

### 3.1. Assumptions on the Noise

For the sake of simplicity, some simplifying assumptions will be done. First we assume that the covariance $\mathscr{Q}$ of the noise is diagonal in the Fourier basis, so that we can write

$$\mathscr{Q}v = \sum_{\mathbf{k}\in\mathbf{Z}^3} (q_{\mathbf{k}}\cdot v_{\mathbf{k}})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}.$$

Moreover we assume that $q_{\mathbf{k}}^T\cdot\mathbf{k} = \mathbf{0}$ for each $\mathbf{k}$, this implies that the Brownian motion takes values in the space of divergence-free vector fields. The Brownian motion has finite variance that we denote by $\sigma^2$. We assume also that for each $\mathbf{k}$, the real and the imaginary parts of the $3\times 3$ matrix $q_{\mathbf{k}}$, if not zero, have rank 2. This is an assumption *in the small* of non-degeneracy, since we ask that, if a mode is forced, it is fully forced in its 4 components. As a first consequence of our assumptions, the operators $\mathscr{Q}$ and $\mathscr{P}$ commute.

The main assumption of the paper is that the noise acts only on a few components, namely most of the matrices $q_{\mathbf{k}}$ are zero. We define the set $\mathscr{N}\subset\mathbf{Z}^3$ of stochastically forced indices, that is the set of $\mathbf{k}$s such that $q_{\mathbf{k}}\not\equiv 0$.

### 3.2. The Equation in the Fourier Modes

We write

$$u(t, x) = \sum_{\mathbf{k} \in \mathbf{Z}^3} u_{\mathbf{k}}(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x}$$

and, by means of the operator $\mathscr{P}$, we project the equations in the space of divergence-free vector fields, so that the pressure disappears. We obtain the following infinite system of stochastic differential equations (see also Gallavotti,[6] Chapter 2, where the author gives also a interpretation of the physics of the fluid in terms of the Fourier coordinates)

$$du_{\mathbf{k}} = \left[ -\nu\, |\mathbf{k}|^2\, u_{\mathbf{k}} - \mathrm{i} \sum_{\substack{\mathbf{h}, \mathbf{l} \in \mathbf{Z}^3 \\ \mathbf{h}+\mathbf{l}=\mathbf{k}}} (\mathbf{k} \cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k} \cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\, \mathbf{k} \right) \right] dt + q_{\mathbf{k}}\, d\beta_t^{\mathbf{k}},$$

$$u_{\mathbf{k}} \cdot \mathbf{k} = 0$$

where $(\beta_t^{\mathbf{k}})_{t \geqslant 0}$ are independent three-dimensional Brownian motions, and the nonlinear term has been obtained in the following way:

$$\begin{aligned} \mathscr{P}(u \cdot \nabla)\, u &= \mathrm{i}\mathscr{P} \sum_{\mathbf{k} \in \mathbf{Z}^3} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (\mathbf{l} \cdot u_{\mathbf{h}})\, u_{\mathbf{l}} \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x} \\ &= \mathrm{i} \sum_{\mathbf{k} \in \mathbf{Z}^3} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (\mathbf{l} \cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k} \cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\, \mathbf{k} \right) \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x} \\ &= \mathrm{i} \sum_{\mathbf{k} \in \mathbf{Z}^3} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (\mathbf{k} \cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k} \cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\, \mathbf{k} \right) \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x}. \end{aligned}$$

### 3.3. The Finite Dimensional Approximation

Let $N \in \mathbf{N}$ and set

$$\mathscr{K}_N = \{\mathbf{k} \in \mathbf{Z}^3 \mid \mathbf{k} \neq (0, 0, 0),\, |\mathbf{k}|_\infty \leqslant N\}.$$

where $|\cdot|_\infty$ is the sup-norm in $\mathbf{R}^3$. We project the equation in the space spanned by $(\mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x})_{\mathbf{k} \in \mathscr{K}_N}$, with coefficients in $\mathbf{R}^3$, and to this end we set

$$u(t, x) = \sum_{\mathbf{k} \in \mathscr{K}_N} u_{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot x}.$$

The equations in the finite dimensional approximation are

$$du_{\mathbf{k}} = \left[ -\nu|\mathbf{k}|^2 u_{\mathbf{k}} - \mathrm{i} \sum_{\substack{\mathbf{h},\mathbf{l} \in \mathscr{K}_N \\ \mathbf{h}+\mathbf{l}=\mathbf{k}}} (\mathbf{k}\cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\mathbf{k} \right) \right] dt + q_{\mathbf{k}}\, d\beta_t^{\mathbf{k}},$$

with $\mathbf{k} \in \mathscr{K}_N$. We set

$$u_{\mathbf{k}} = (r_{\mathbf{k}}^j + \mathrm{i} s_{\mathbf{k}}^j)_{j=1,2,3},$$

where $\mathbf{k}\cdot r_{\mathbf{k}} = \mathbf{k}\cdot s_{\mathbf{k}} = 0$ and $r_{\mathbf{k}}^j$, $s_{\mathbf{k}}^j$, $j=1,2,3$, are real-valued. Since $u_{-\mathbf{k}} = \overline{u_{\mathbf{k}}}$, we are going to choose a smaller set of indices $\mathbf{k} \in \mathscr{K}_N$ in order to take into account that some equations in the system are redundant. We set

$$\mathscr{K}_N^1 = \{\mathbf{k} \in \mathbf{Z}^3 \mid |\mathbf{k}|_\infty \leqslant N,\, k_3 > 0\}$$
$$\mathscr{K}_N^2 = \{\mathbf{k} \in \mathbf{Z}^3 \mid |\mathbf{k}|_\infty \leqslant N,\, k_3 = 0,\, k_2 > 0\}$$
$$\mathscr{K}_N^3 = \{\mathbf{k} \in \mathbf{Z}^3 \mid |\mathbf{k}|_\infty \leqslant N,\, k_3 = k_2 = 0,\, k_1 > 0\}$$

and

$$\widetilde{\mathscr{K}} = \mathscr{K}_N^1 \cup \mathscr{K}_N^2 \cup \mathscr{K}_N^3,$$

in such a way that

$$\mathscr{K}_N = \widetilde{\mathscr{K}} \cup (-\widetilde{\mathscr{K}}) \qquad \text{and} \qquad \widetilde{\mathscr{K}} \cap (-\widetilde{\mathscr{K}}) = \varnothing.$$

Notice that $\#(\widetilde{\mathscr{K}}) = \frac{1}{2}[(2N+1)^3 - 1]$, we call such number $D$. Now, if $\mathbf{k} \in \widetilde{\mathscr{K}}$, the sum extended to all pairs of indices $\mathbf{h}$, $\mathbf{l}$ such that $\mathbf{h}+\mathbf{l}=\mathbf{k}$ can be written in the following way:

$$\sum_{\substack{\mathbf{h}+\mathbf{l}=\mathbf{k} \\ \mathbf{h},\mathbf{l}\in\mathscr{K}_N}} = \sum_{\substack{\mathbf{h}+\mathbf{l}=\mathbf{k} \\ \mathbf{h},\mathbf{l}\in\widetilde{\mathscr{K}}}} + \sum_{\substack{\mathbf{h}+\mathbf{l}=\mathbf{k} \\ \mathbf{h}\in\widetilde{\mathscr{K}} \\ \mathbf{l}\in-\widetilde{\mathscr{K}}}} + \sum_{\substack{\mathbf{h}+\mathbf{l}=\mathbf{k} \\ \mathbf{h}\in-\widetilde{\mathscr{K}} \\ \mathbf{l}\in\widetilde{\mathscr{K}}}},$$

since if $\mathbf{h}, \mathbf{l} \notin \widetilde{\mathscr{K}}$, $\mathbf{k}$ does not belong to $\widetilde{\mathscr{K}}$ as well. We denote by $\sum^*$ the sum extended to indices in $\widetilde{\mathscr{K}}$. With this position

$$\sum_{\substack{\mathbf{h}+\mathbf{l}=\mathbf{k} \\ \mathbf{h},\mathbf{l}\in\mathscr{K}_N}} (\mathbf{k}\cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\mathbf{k} \right)$$
$$= \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}^* (\mathbf{k}\cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\mathbf{k} \right) + \sum_{\mathbf{h}-\mathbf{l}=\mathbf{k}}^* (\mathbf{k}\cdot u_{\mathbf{h}}) \left( \overline{u_{\mathbf{l}}} - \frac{\mathbf{k}\cdot \overline{u_{\mathbf{l}}}}{|\mathbf{k}|^2}\mathbf{k} \right)$$
$$+ \sum_{\mathbf{l}-\mathbf{h}=\mathbf{k}}^* (\mathbf{k}\cdot \overline{u_{\mathbf{h}}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\mathbf{k} \right)$$

so that the equations become

$$du_{\mathbf{k}}+\Bigg(\nu\,|\mathbf{k}|^2\,u_{\mathbf{k}}+\mathrm{i}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot u_{\mathbf{h}})\Big(u_{\mathbf{l}}-\frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\,\mathbf{k}\Big)$$

$$+\mathrm{i}\sum_{\mathbf{h}-\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot u_{\mathbf{h}})\Big(\overline{u_{\mathbf{l}}}-\frac{\mathbf{k}\cdot\overline{u_{\mathbf{l}}}}{|\mathbf{k}|^2}\,\mathbf{k}\Big)$$

$$+\mathrm{i}\sum_{\mathbf{l}-\mathbf{h}=\mathbf{k}}^{*}(\mathbf{k}\cdot\overline{u_{\mathbf{h}}})\Big(u_{\mathbf{l}}-\frac{\mathbf{k}\cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\,\mathbf{k}\Big)\Bigg)dt=q_{\mathbf{k}}\,d\beta_t^{\mathbf{k}}.$$

It is convenient to write explicitly the equations relative to the real and imaginary part of $u_{\mathbf{k}}$. To this end, we set

$$F_{r_{\mathbf{k}}^i}=-\nu\,|\mathbf{k}|^2\,r_{\mathbf{k}}^i+\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)+(\mathbf{k}\cdot s_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)$$

$$-\sum_{\mathbf{h}-\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)-(\mathbf{k}\cdot s_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)$$

$$+\sum_{\mathbf{l}-\mathbf{h}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)-(\mathbf{k}\cdot s_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big) \tag{3.1}$$

and

$$F_{s_{\mathbf{k}}^i}=-\nu\,|\mathbf{k}|^2\,s_{\mathbf{k}}^i-\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)-(\mathbf{k}\cdot s_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)$$

$$-\sum_{\mathbf{h}-\mathbf{l}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)+(\mathbf{k}\cdot s_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)$$

$$-\sum_{\mathbf{l}-\mathbf{h}=\mathbf{k}}^{*}(\mathbf{k}\cdot r_{\mathbf{h}})\Big(r_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot r_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big)+(\mathbf{k}\cdot s_{\mathbf{h}})\Big(s_{\mathbf{l}}^i-\frac{\mathbf{k}\cdot s_{\mathbf{l}}}{|\mathbf{k}|^2}\,k_i\Big), \tag{3.2}$$

so that the equations for the real and the imaginary part become

$$dr_{\mathbf{k}}-F_{r_{\mathbf{k}}}(r,s)\,dt=q_{\mathbf{k}}^r\,d\beta_t^{\mathbf{k}}$$

and

$$ds_{\mathbf{k}}-F_{s_{\mathbf{k}}}(r,s)\,dt=q_{\mathbf{k}}^s\,d\beta_t^{\mathbf{k}}.$$

## 4. THE LIE ALGEBRA GENERATED BY THE DYNAMICS

The state space of the Markov process $(r(t), s(t))$ which is solution of the equations stated above is a linear space $U \subset \mathbf{R}^{6D}$, where $D = \#\tilde{\mathscr{K}}$, given by

$$U = \bigoplus_{\mathbf{k} \in \tilde{\mathscr{K}}} (R_{\mathbf{k}} \oplus S_{\mathbf{k}}), \tag{4.1}$$

and each element of $U$ is labelled $(r, s)$, with $r = (r_{\mathbf{k}}^1, r_{\mathbf{k}}^2, r_{\mathbf{k}}^3)_{\mathbf{k} \in \tilde{\mathscr{K}}}$ and $s = (s_{\mathbf{k}}^1, s_{\mathbf{k}}^2, s_{\mathbf{k}}^3)_{\mathbf{k} \in \tilde{\mathscr{K}}}$, and

$$R_{\mathbf{k}} = \{(r, s) \in \mathbf{R}^{6D} \mid r_{\mathbf{k}} \cdot \mathbf{k} = 0, s_{\mathbf{k}} = 0, r_{\mathbf{h}} = s_{\mathbf{h}} = 0, \mathbf{h} \neq \mathbf{k}\}$$
$$S_{\mathbf{k}} = \{(r, s) \in \mathbf{R}^{6D} \mid s_{\mathbf{k}} \cdot \mathbf{k} = 0, r_{\mathbf{k}} = 0, r_{\mathbf{h}} = s_{\mathbf{h}} = 0, \mathbf{h} \neq \mathbf{k}\}.$$

In the same way, we can define the Lie algebra $\mathfrak{U}$ corresponding to the vector space $U$,

$$\mathfrak{U} = \left\{ G \mid G = \sum_{\substack{\mathbf{k} \in \tilde{\mathscr{K}} \\ i = 1, 2, 3}} G_{r_{\mathbf{k}}^i} \frac{\partial}{\partial r_{\mathbf{k}}^i} + G_{s_{\mathbf{k}}^i} \frac{\partial}{\partial s_{\mathbf{k}}^i} \text{ and } \mathbf{k} \cdot G_{r_{\mathbf{k}}} = \mathbf{k} \cdot G_{s_{\mathbf{k}}} = 0 \right\}. \tag{4.2}$$

We define also the subspaces $\mathfrak{U}_{\mathbf{k}} = \mathfrak{R}_{\mathbf{k}} \oplus \mathfrak{S}_{\mathbf{k}}$ of $\mathfrak{U}$ of constant vector fields, where

$$\mathfrak{R}_{\mathbf{k}} = \left\{ \sum_{i = 1, 2, 3} r_{\mathbf{k}}^i \frac{\partial}{\partial r_{\mathbf{k}}^i} \,\middle|\, r_{\mathbf{k}} \in R_{\mathbf{k}} \right\} \quad \text{and} \quad \mathfrak{S}_{\mathbf{k}} = \left\{ \sum_{i = 1, 2, 3} s_{\mathbf{k}}^i \frac{\partial}{\partial s_{\mathbf{k}}^i} \,\middle|\, s_{\mathbf{k}} \in S_{\mathbf{k}} \right\}$$

In this section, we want to find some reasonable conditions on the set $\mathscr{N}$ of forced modes (such a set has been defined in Section 3.1) in such a way that the algebra generated by the fields

$$\{F_0\} \cup \mathfrak{U}_{\mathbf{k}} \qquad \mathbf{k} \in \mathscr{N}, \tag{4.3}$$

where

$$F_0 = \sum_{\substack{\mathbf{k} \in \tilde{\mathscr{K}} \\ i = 1, 2, 3}} F_{r_{\mathbf{k}}^i} \frac{\partial}{\partial r_{\mathbf{k}}^i} + F_{s_{\mathbf{k}}^i} \frac{\partial}{\partial s_{\mathbf{k}}^i},$$

and $F_{r^i_{\mathbf{k}}}$ and $F_{s^i_{\mathbf{k}}}$ have been defined respectively in (3.1) and (3.2), contains all the constant vector fields of $\mathfrak{U}$. In particular, it follows that the Hörmander condition holds, that is the generated Lie algebra, evaluated at each point of $U$, gives $U$ itself. We start with some computations that will be useful in the sequel.

**Lemma 4.1.** Let $\mathbf{m}, \mathbf{n} \in \tilde{\mathscr{K}}$ and $V \in \mathfrak{U}_{\mathbf{m}}$, $W \in \mathfrak{U}_{\mathbf{n}}$, with

$$V = \sum_{j=1}^{3} v^r_j \frac{\partial}{\partial r^j_{\mathbf{m}}} + v^s_j \frac{\partial}{\partial s^j_{\mathbf{m}}}, \qquad W = \sum_{j=1}^{3} w^r_j \frac{\partial}{\partial r^j_{\mathbf{m}}} + w^s_j \frac{\partial}{\partial s^j_{\mathbf{m}}},$$

then

(i) if $\mathbf{k} = \mathbf{m} + \mathbf{n}$, $\mathbf{h} = \mathbf{n} - \mathbf{m}$ and $\mathbf{g} = \mathbf{m} - \mathbf{n}$, the vector field $[[F_0, V], W]$ is equal to

$$\begin{aligned}
&[(v^r \cdot \mathbf{k})\, P_{\mathbf{k}}(w^s) + (w^s \cdot \mathbf{k})\, P_{\mathbf{k}}(v^r) + (v^s \cdot \mathbf{k})\, P_{\mathbf{k}}(w^r) + (w^r \cdot \mathbf{k})\, P_{\mathbf{k}}(v^s)] \frac{\partial}{\partial r_{\mathbf{k}}} \\
&\quad + [(v^s \cdot \mathbf{k})\, P_{\mathbf{k}}(w^s) + (w^s \cdot \mathbf{k})\, P_{\mathbf{k}}(v^s) - (v^r \cdot \mathbf{k})\, P_{\mathbf{k}}(w^r) - (w^r \cdot \mathbf{k})\, P_{\mathbf{k}}(v^r)] \frac{\partial}{\partial s_{\mathbf{k}}} \\
&\quad + [(v^r \cdot \mathbf{h})\, P_{\mathbf{h}}(w^s) + (w^s \cdot \mathbf{h})\, P_{\mathbf{h}}(v^r) - (v^s \cdot \mathbf{h})\, P_{\mathbf{h}}(w^r) - (w^r \cdot \mathbf{h})\, P_{\mathbf{h}}(v^s)] \frac{\partial}{\partial r_{\mathbf{h}}} \\
&\quad - [(v^r \cdot \mathbf{h})\, P_{\mathbf{h}}(w^r) + (w^r \cdot \mathbf{h})\, P_{\mathbf{h}}(v^r) + (v^s \cdot \mathbf{h})\, P_{\mathbf{h}}(w^s) + (w^s \cdot \mathbf{h})\, P_{\mathbf{h}}(v^s)] \frac{\partial}{\partial s_{\mathbf{h}}} \\
&\quad + [(v^s \cdot \mathbf{g})\, P_{\mathbf{g}}(w^r) + (w^r \cdot \mathbf{g})\, P_{\mathbf{g}}(v^s) - (v^r \cdot \mathbf{g})\, P_{\mathbf{g}}(w^s) - (w^s \cdot \mathbf{g})\, P_{\mathbf{g}}(v^r)] \frac{\partial}{\partial r_{\mathbf{g}}} \\
&\quad - [(v^r \cdot \mathbf{g})\, P_{\mathbf{g}}(w^r) + (w^r \cdot \mathbf{g})\, P_{\mathbf{g}}(v^r) + (v^s \cdot \mathbf{g})\, P_{\mathbf{g}}(w^s) + (w^s \cdot \mathbf{g})\, P_{\mathbf{g}}(v^s)] \frac{\partial}{\partial s_{\mathbf{g}}},
\end{aligned}$$

where $P_{\mathbf{k}}$ is the projection of $\mathbf{R}^3$ on the plane orthogonal to the vector $\mathbf{k}$, and in the above formula the terms corresponding to indices out of $\tilde{\mathscr{K}}$ are zero;

(ii) if there is $q \in \mathbf{Q}$ such that $\mathbf{n} = q\,\mathbf{m}$, then $[[F_0, V], W] = 0$,

(iii) $[[F_0, V], W] = \frac{1}{2}[[F_0, V + W], V + W]$.

*Proof.* We compute the derivatives of the components of $F_0$ (defined in (3.1) and (3.2)),

$$\frac{\partial F_{r^i_{\mathbf{k}}}}{\partial r^j_{\mathbf{m}}} = -\nu\,|\mathbf{k}|^2\,\delta_{ij}\delta_{\mathbf{km}} + k_j(s^i_{\mathbf{k}-\mathbf{m}} - s^i_{\mathbf{m}-\mathbf{k}} + s^i_{\mathbf{m}+\mathbf{k}}) + \mathbf{k}\cdot(s_{\mathbf{k}-\mathbf{m}} - s_{\mathbf{m}-\mathbf{k}} + s_{\mathbf{m}+\mathbf{k}})\,\alpha_{ij}(\mathbf{k})$$

$$\frac{\partial F_{r^i_{\mathbf{k}}}}{\partial s^j_{\mathbf{m}}} = k_j(r^i_{\mathbf{k}-\mathbf{m}} + r^i_{\mathbf{m}-\mathbf{k}} - r^i_{\mathbf{m}+\mathbf{k}}) + \mathbf{k}\cdot(r_{\mathbf{k}-\mathbf{m}} + r_{\mathbf{m}-\mathbf{k}} - r_{\mathbf{m}+\mathbf{k}})\,\alpha_{ij}(\mathbf{k})$$

$$\frac{\partial F_{s^i_{\mathbf{k}}}}{\partial r^j_{\mathbf{m}}} = -k_j(r^i_{\mathbf{k}-\mathbf{m}} + r^i_{\mathbf{m}-\mathbf{k}} + r^i_{\mathbf{m}+\mathbf{k}}) - \mathbf{k}\cdot(r_{\mathbf{k}-\mathbf{m}} + r_{\mathbf{m}-\mathbf{k}} + r_{\mathbf{m}+\mathbf{k}})\,\alpha_{ij}(\mathbf{k})$$

$$\frac{\partial F_{s^i_{\mathbf{k}}}}{\partial s^j_{\mathbf{m}}} = -\nu\,|\mathbf{k}|^2\,\delta_{ij}\delta_{\mathbf{km}} + k_j(s^i_{\mathbf{k}-\mathbf{m}} - s^i_{\mathbf{m}-\mathbf{k}} - s^i_{\mathbf{m}+\mathbf{k}}) + \mathbf{k}\cdot(s_{\mathbf{k}-\mathbf{m}} - s_{\mathbf{m}-\mathbf{k}} - s_{\mathbf{m}+\mathbf{k}})\,\alpha_{ij}(\mathbf{k})$$

where we have set, for brevity, $\alpha_{ij}(\mathbf{k}) = \delta_{ij} - \frac{2k_ik_j}{|\mathbf{k}|^2}$, and the second derivatives

$$\frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial r^l_{\mathbf{n}}\,\partial r^j_{\mathbf{m}}} = \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial s^l_{\mathbf{n}}\,\partial s^j_{\mathbf{m}}} = 0, \qquad \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial s^l_{\mathbf{n}}\,\partial r^j_{\mathbf{m}}} = (\delta_{\mathbf{n},\,\mathbf{k}-\mathbf{m}} - \delta_{\mathbf{n},\,\mathbf{m}-\mathbf{k}} + \delta_{\mathbf{n},\,\mathbf{m}+\mathbf{k}})\,A^i_{ij}(\mathbf{k})$$

and

$$\frac{\partial^2 F_{s^i_{\mathbf{k}}}}{\partial r^l_{\mathbf{n}}\,\partial r^j_{\mathbf{m}}} = -(\delta_{\mathbf{n},\,\mathbf{k}-\mathbf{m}} + \delta_{\mathbf{n},\,\mathbf{m}-\mathbf{k}} + \delta_{\mathbf{n},\,\mathbf{m}+\mathbf{k}})\,A^i_{jl}(\mathbf{k}), \qquad \frac{\partial^2 F_{s^i_{\mathbf{k}}}}{\partial s^l_{\mathbf{n}}\,\partial r^j_{\mathbf{m}}} = 0$$

$$\frac{\partial^2 F_{s^i_{\mathbf{k}}}}{\partial s^l_{\mathbf{n}}\,\partial s^j_{\mathbf{m}}} = (\delta_{\mathbf{n},\,\mathbf{k}-\mathbf{m}} - \delta_{\mathbf{n},\,\mathbf{m}-\mathbf{k}} - \delta_{\mathbf{n},\,\mathbf{m}+\mathbf{k}})\,A^i_{jl}(\mathbf{k})$$

(we have set $A^i_{jl}(\mathbf{k}) = \delta_{il}k_j + \delta_{ij}k_l - 2\,\frac{k_ik_jk_l}{|\mathbf{k}|^2}$), with the agreement that everything concerning indices out of the set $\widetilde{\mathscr{K}}$ is zero. Take now $V \in \mathfrak{U}_{\mathbf{m}}$ and $W \in \mathfrak{U}_{\mathbf{n}}$ as in the statement of the lemma, then by computing the bracket we obtain

$$[[F_0, V], W] = \sum_{\mathbf{k}\in\widetilde{\mathscr{K}}} \sum_{i,j,l=1}^{3} \left( v^s_j w^r_l \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial s^j_{\mathbf{m}}\,\partial r^l_{\mathbf{n}}} + v^r_j w^s_l \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial r^j_{\mathbf{m}}\,\partial s^l_{\mathbf{n}}} \right) \frac{\partial}{\partial r^i_{\mathbf{k}}} + \left( v^r_j w^r_l \frac{\partial^2 F_{s^i_{\mathbf{k}}}}{\partial r^j_{\mathbf{m}}\,\partial r^l_{\mathbf{n}}} + v^s_j w^s_l \frac{\partial^2 F_{s^i_{\mathbf{k}}}}{\partial s^j_{\mathbf{m}}\,\partial s^l_{\mathbf{n}}} \right) \frac{\partial}{\partial s^i_{\mathbf{k}}}.$$

We analyse the coefficients of the $\partial_{r^i_{\mathbf{k}}}$-components:

$$v^s_j w^r_l \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial s^j_{\mathbf{m}}\, \partial r^l_{\mathbf{n}}} + v^r_j w^s_l \frac{\partial^2 F_{r^i_{\mathbf{k}}}}{\partial r^j_{\mathbf{m}}\, \partial s^l_{\mathbf{n}}}$$

$$= \sum_{j,l=1}^{3} (\delta_{\mathbf{m},\,\mathbf{k}-\mathbf{n}} - \delta_{\mathbf{m},\,\mathbf{n}-\mathbf{k}} + \delta_{\mathbf{m},\,\mathbf{n}+\mathbf{k}})\, A^i_{jl}(\mathbf{k})\, v^s_j w^r_l$$

$$+ (\delta_{\mathbf{n},\,\mathbf{k}-\mathbf{m}} - \delta_{\mathbf{n},\,\mathbf{m}-\mathbf{k}} + \delta_{\mathbf{n},\,\mathbf{m}+\mathbf{k}})\, A^i_{jl}(\mathbf{k})\, v^r_j w^s_l$$

$$= (\delta_{\mathbf{m},\,\mathbf{k}-\mathbf{n}} - \delta_{\mathbf{m},\,\mathbf{n}-\mathbf{k}} + \delta_{\mathbf{m},\,\mathbf{n}+\mathbf{k}})[(v^s\cdot\mathbf{k})\, P_{\mathbf{k}}(w^r)_i + (w^r\cdot\mathbf{k})\, P_{\mathbf{k}}(v^s)_i]$$

$$+ (\delta_{\mathbf{n},\,\mathbf{k}-\mathbf{m}} - \delta_{\mathbf{n},\,\mathbf{m}-\mathbf{k}} + \delta_{\mathbf{n},\,\mathbf{m}+\mathbf{k}})[(v^r\cdot\mathbf{k})\, P_{\mathbf{k}}(w^s)_i + (w^s\cdot\mathbf{k})\, P_{\mathbf{k}}(v^r)_i],$$

where $P_{\mathbf{k}}(v)_i = v_i - \frac{k_i}{|\mathbf{k}|^2}(v\cdot\mathbf{k})$. In a similar way it is possible to treat the coefficients of the $\partial_{s^i_{\mathbf{k}}}$-components, and claim (i) is true.

If $\mathbf{n} = q\,\mathbf{m}$, it follows that

$$v^r\cdot\mathbf{k} = v^s\cdot\mathbf{k} = w^r\cdot\mathbf{k} = w^s\cdot\mathbf{k} = 0$$

with $\mathbf{k} = \mathbf{m}+\mathbf{n}$, $\mathbf{m}-\mathbf{n}$ and $\mathbf{n}-\mathbf{m}$, so that using property (i) of this lemma, claim (ii) holds true. Finally, if $V \in \mathfrak{U}_{\mathbf{m}}$ and $W \in \mathfrak{U}_{\mathbf{n}}$,

$$[[F_0, V+W], V+W]$$

$$= [[F_0, V], V] + [[F_0, V], W] + [[F_0, W], V] + [[F_0, W], W]$$

which, by the Jacobi identity and by property (ii), is equal to $2[[F_0, V], W]$. ∎

The computations of the above lemma show that the non-linear term mixes and combines the components. In some sense, this mechanism can be considered as a geometrical counterpart of the cascade of energy. Our aim is to understand for which sets $\mathscr{N}$ of forced modes the evaluation of the Lie algebra generated by the fields (4.3), gives $U$. We define the set $A(\mathscr{N}) \subset \mathscr{K}_N$ of indices $\mathbf{k} \in K_N$ such that the constant vector fields corresponding to $\mathbf{k}$ (or to $-\mathbf{k}$, depending on $\mathbf{k} \in \widetilde{\mathscr{K}}$ or $-\mathbf{k} \in \widetilde{\mathscr{K}}$) are in the Lie algebra generated by the vector fields (4.3). Obviously, $\mathscr{N} \subset A(\mathscr{N})$, and our aim is to show that $A(\mathscr{N}) = \mathscr{K}_N$.

**Lemma 4.2.** Let $\mathscr{N}$ be a subset of indices and define the set $A(\mathscr{N})$ as above.

(i) If $\mathbf{m} \in A(\mathscr{N})$, then also $-\mathbf{m} \in A(\mathscr{N})$,

(ii) if $\mathbf{m}$, $\mathbf{n}$ are in $A(\mathscr{N})$, $\mathbf{m}+\mathbf{n}$ is in $\mathscr{K}_N$, $\mathbf{m}$ and $\mathbf{n}$ are linearly independent and $|\mathbf{m}| \neq |\mathbf{n}|$, then $\mathbf{m}+\mathbf{n} \in A(\mathscr{N})$.

*Proof.* The first property follows from the fact that $u_{-\mathbf{m}} = \overline{u_{\mathbf{m}}}$. In order to show the second claim, take $\mathbf{m}$ and $\mathbf{n}$ in $A(\mathcal{N}) \cap \widetilde{\mathcal{K}}$ and assume that $\mathbf{k} = \mathbf{m} + \mathbf{n} \in \widetilde{\mathcal{K}}$. The claim follows if $\mathbf{m} + \mathbf{n} \in A(\mathcal{N})$.

Let

$$V^r = \sum_{i=1}^{3} v_i \frac{\partial}{\partial r^i_{\mathbf{m}}}, \qquad V^s = \sum_{i=1}^{3} v_i \frac{\partial}{\partial s^i_{\mathbf{m}}},$$

$$W^r = \sum_{i=1}^{3} w_i \frac{\partial}{\partial r^i_{\mathbf{n}}}, \qquad W^s = \sum_{i=1}^{3} w_i \frac{\partial}{\partial s^i_{\mathbf{n}}},$$

with $v \cdot \mathbf{m} = w \cdot \mathbf{n} = 0$. Then, by property (i) of the previous lemma,

$$[[F_0, V^r], W^s] + [[F_0, V^s], W^r] = 2((v \cdot \mathbf{k})\, P_{\mathbf{k}}(w) + (w \cdot \mathbf{k})\, P_{\mathbf{k}}(v)) \cdot \frac{\partial}{\partial r_{\mathbf{k}}},$$

$$[[F_0, V^r], W^r] - [[F_0, V^s], W^s] = 2((v \cdot \mathbf{k})\, P_{\mathbf{k}}(w) + (w \cdot \mathbf{k})\, P_{\mathbf{k}}(v)) \cdot \frac{\partial}{\partial s_{\mathbf{k}}}.$$

Now, let $M$, $E$ two vectors in $\mathbf{R}^3$ such that $\{\mathbf{k}, M, E\}$ is a basis of $\mathbf{R}^3$, $M$ and $E$ span $\{x \in \mathbf{R}^3 \mid x \cdot \mathbf{k} = 0\}$ and $\mathbf{m}$, $\mathbf{n}$ are in Span$[\mathbf{k}, M]$. Choose

$$v = \lambda_1 \mathbf{k} + \mu_1 M + \nu_1 E, \qquad w = \lambda_2 \mathbf{k} + \mu_2 M + \nu_2 E,$$

then, by the assumptions on $\mathbf{m}$ and $\mathbf{n}$, it is always possible to choose the coefficients $\lambda_1$, $\mu_1$, $\nu_1$, $\lambda_2$, $\mu_2$, $\nu_2$ in such a way that $(v \cdot \mathbf{k})\, P_{\mathbf{k}}(w) + (w \cdot \mathbf{k})\, P_{\mathbf{k}}(v)$ is any vector in Span$[M, E]$. In other words, $\mathfrak{U}_{\mathbf{k}}$ is contained in the Lie algebra generated by the vector fields (4.3). In the same way, if $\mathbf{h} = \mathbf{n} - \mathbf{m} \in \widetilde{\mathcal{K}}$ (or if $\mathbf{g} = \mathbf{m} - \mathbf{n} \in \widetilde{\mathcal{K}}$), the conclusion follows by taking $[[F_0, V^r], W^s] - [[F_0, V^s], W^r]$ and $[[F_0, V^r], W^r] + [[F_0, V^s], W^s]$. ∎

## 5. DETERMINING SETS OF INDICES

In view of Lemma 4.2, we call a subset $\mathcal{N}$ of $\mathcal{K}_N$ a *determining set of indices* for the ultraviolet cut-off $N$, if $\mathcal{N}$ generates the cube $K_N$ in the sense that $A(\mathcal{N}) = \mathcal{K}_N$, where $A(\mathcal{N})$ has been defined in the previous section. Lemma 4.2 shows us which is the algebraic structure of such set. Namely, $A(\mathcal{N})$ is symmetric with respect to the origin and it is close with respect to the sum, under some restrictions ($\mathbf{m}$ and $\mathbf{n}$ have to be linearly independent, with $|\mathbf{m}| \neq |\mathbf{n}|$ and $\mathbf{m} + \mathbf{n} \in \mathcal{K}_N$). If one neglects such restrictions, Lemma 4.2 tells us that a set $\mathcal{N}$ is a determining set of indices for the cut-off $N$ if it is an algebraic system of generators for the group $(\mathbf{Z}^3, +)$, that is, the smallest subgroup of $\mathbf{Z}^3$ which contains $\mathcal{N}$ is the whole $\mathbf{Z}^3$.

Since by Lemma 4.2 it is obviously true that a determining set of indices, with respect to any cut-off, is a system of generators, one can ask if the vice-versa is true, that is if each system of generators is a determining set of indices for a suitable cut-off. We give this statement in the form of a claim, since in our opinion any proof seems to be full of technicalities which are not of great interest in this context.

**Claim 5.1.** If $\mathscr{N}$ is an algebraic system of generators for the group $(\mathbf{Z}^3, +)$ and $\mathscr{N} \subset \mathscr{K}_N$, then $\mathscr{N}$ is a determining set of indices for the ultraviolet cut-off $N$.

For the sake of completeness, we give (see Jacobson,[8] Theorems 3.8 and 3.9) a necessary and sufficient condition for a set of indices $\mathscr{N}$ to be a system of generators of the whole group $\mathbf{Z}^3$.

**Theorem 5.2.** A set $\mathscr{N} \subset \mathbf{Z}^3$ is a system of generators of $\mathbf{Z}^3$ if and only if the g.c.d. of the minors of order 3 of the matrix $A$ is equal to 1, where $A$ is the $k \times 3$ matrix whose rows are the coordinates of the points of $\mathscr{N}$ and $k = \#\mathscr{N}$.

The intuitive idea which lets us believe that the claim is true is that the restrictions given in the statement of property (ii) of Lemma 4.2 can be avoided in the following way.

The restriction about linear independence can be easily avoided by *moving aside*: for example if one wants to sum $\mathbf{m}$ with itself, the best way is to obtain $2\mathbf{m}$ as $\mathbf{m}+\mathbf{n}+\mathbf{m}-\mathbf{n}$, where $\mathbf{n}$ is linear independent with $\mathbf{m}$.

The restriction about the Euclidean norm (that is, $|\mathbf{m}| \neq |\mathbf{n}|$) can be avoided, where possible, as in the previous case. Sometimes, as in the case of the proposition below, this is not possible, since it may happen that all indices we are allowed to use, have the same Euclidean norm. In such a case the solution is to *reach* the index by different paths, providing with each path a component of the Lie algebra we are dealing with, in analogy with Lemma 4.1. This method is probably peculiar of the dimension three and it does not hold in lower dimensions (see E and Mattingly[3]).

Indeed these tricks are used in the proof of the following proposition, which states that the *working example* we talked about in Section 2 is a determining set of indices.

**Proposition 5.2.** Any set $\mathscr{N} \subset \mathbf{Z}^3$ containing the three indices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is a determining set of indices.

*Proof.* A careful analysis of the last part of the proof of Lemma 4.2 shows that, if $|\mathbf{m}| = |\mathbf{n}|$, then the Lie brackets $[[F_0, V], W]$, with $V \in \mathfrak{U}_{\mathbf{m}}$ and $W \in \mathfrak{U}_{\mathbf{n}}$, span the two-dimensional subspace of $\mathfrak{U}_{\mathbf{m}+\mathbf{n}}$ given by

$$\lambda E \cdot \frac{\partial}{\partial r_{\mathbf{m}+\mathbf{n}}} + \mu E \cdot \frac{\partial}{\partial s_{\mathbf{m}+\mathbf{n}}},$$

where $E$ is the index orthogonal (in $\mathbf{R}^3$) to $\mathbf{m}$ and $\mathbf{n}$. Hence, if we sum $(1, 0, 0)$ and $(0, 1, 0)$, we obtain the corresponding two dimensional subspace of $\mathfrak{U}_{(1,1,0)}$. Again a direct computation shows that such a smaller subspace is indeed sufficient, since if we combine it with $\mathfrak{U}_{(0,0,1)}$ we obtain the full $\mathfrak{U}_{(1,1,1)}$. Now we just subtract $(0, 0, 1)$ from $(1, 1, 1)$ to obtain the full $\mathfrak{U}_{(1,1,0)}$ and, in the same way, we can obtain all the indices of norm 2. With this set of indices is now easy to obtain, by means of Lemma 4.2 and of the tricks explained above, all the indices in $\mathscr{K}_N$, whatever is $N$. ∎

An obvious consequence of the above proposition is that if $\mathscr{N}$ is a determining set of indices for a cut-off $N$, then it is a determining set of indices for any other cut-off threshold larger than $N$.

## 6. THE CONTROL PROBLEM

The section is devoted to the proof of the controllability properties of the finite dimensional approximations of Navier–Stokes equations. The first part contains some generalities on polynomial control systems. The approach and the results are taken from Jurdjevic and Kupka.[10] In the second part we adapt the proof of a theorem (again of Jurdjevic and Kupka[10]) to our case. The original theorem applies to polynomials of odd degree. Polynomials of even degree behave in a different way, mostly because of the *obstructions* of the positive terms. Our case has no obstructions, essentially because of property (ii) of Lemma 4.1, and the system is controllable.

### 6.1. Generalities on Polynomial Control Systems

We consider a system of the form

$$\dot{x} = P(x) + \sum_{i=1}^{m} u_i(t)$$

where $x \in \mathbf{R}^n$, $b_1, b_2, \ldots, b_n$ are fixed vectors in $\mathbf{R}^n$ and $P$ is a polynomial mapping, that is $P = (P_1, \ldots, P_n)$ and each $P_i$ is a polynomial in the variables $(x_1, \ldots, x_n)$. Let $Y_1, \ldots, Y_n$ be the constant vector fields assuming respectively value $b_1,\ldots, b_n$ and let $F$ be the vector fields having the components of $P$ as its components, and define

$$\mathscr{F} = \left\{ F + \sum_{i=1^m} u_i Y_i \mid (u_1,\ldots, u_m) \in \mathbf{R}^m \right\}$$

We define, for each $x_0 \in \mathbf{R}^n$ and $t > 0$, the set $A_{\mathscr{F}}(x_0, t)$ of states reachable, with a suitable control $u = (u_1,\ldots, u_n)$, from the initial state $x_0$ in a time smaller than $t$. We define the set $A^*_{\mathscr{F}}(x_0, t)$ of states reachable exactly at time $t$.

Two families of vector fields $\mathscr{F}_1$, $\mathscr{F}_2$ are said to be equivalent if for all $x \in \mathbf{R}^n$ and $t > 0$,

$$\overline{A_{\mathscr{F}_1}(x, t)} = \overline{A_{\mathscr{F}_2}(x, t)}.$$

If $\mathscr{F}$ is equivalent to $\mathscr{F}_1$ and to $\mathscr{F}_2$, then it is equivalent to $\mathscr{F}_1 \cup \mathscr{F}_2$. It makes sense then to define the saturate of $\mathscr{F}$, denoted by Sat($\mathscr{F}$), which is the union of all families of vector fields equivalent to $\mathscr{F}$. Moreover we will call Lie($\mathscr{F}$) the Lie algebra generated by $\mathscr{F}$. Finally, the *Lie saturate* of $\mathscr{F}$ is defined as LS($\mathscr{F}$) = Sat($\mathscr{F}$) $\cap$ Lie($\mathscr{F}$). In order to obtain controllability, the Lie saturate should be as large as possible, as stated by the following theorem.

**Theorem 6.1.** Let $\mathscr{F}$ be any family of smooth vector fields and assume that LS($\mathscr{F}$) contains $n$ vectors $V_1,\ldots, V_n$ such that the vector space spanned by them is in LS($\mathscr{F}$) and for each $x \in \mathbf{R}^n$ the vectors $V_1(x),\ldots, V_n(x)$ span $\mathbf{R}^n$. Then $A_{\mathscr{F}}(x, t) = \mathbf{R}^n$ for each $x \in \mathbf{R}^n$ and $t > 0$.

We adapt the conclusions of the theorem to the system that will be studied in the following section.

**Corollary 6.2.** Let $\mathscr{F}$ be any family of smooth vector fields and assume that the constant vector fields of LS($\mathscr{F}$) span $\mathbf{R}^n$. Then $A^*_{\mathscr{F}}(x, t) = \mathbf{R}^n$ for each $x \in \mathbf{R}^n$ and $t > 0$.

*Proof.* From the previous theorem, $A_{\mathscr{F}}(x, t) = \mathbf{R}^n$. Moreover, by Theorem 13, Chapter 3 of Jurdjevic[9] (see also the remarks after Theorem 11, Chapter 5 of ref. 9) it follows that also $A^*_{\mathscr{F}}(x, t) = \mathbf{R}^n$. ∎

In the following, we will need the following two lemmata. The first lemma permits the *enlargement* of a family of vector fields by means of diffeomorphisms. A diffeomorphism $\phi\colon \mathbf{R}^n \to \mathbf{R}^n$ is a *normaliser* of a family $\mathscr{F}$ if for all $x \in \mathbf{R}^n$ and $t > 0$,

$$\phi(\overline{A_{\mathscr{F}}(\phi^{-1}(x), t)}) \subset \overline{A_{\mathscr{F}}(x, t)},$$

we will denote by Norm($\mathscr{F}$) the set of all smooth normaliser of $\mathscr{F}$.

**Lemma 6.3.** The family $\mathscr{F}$ is equivalent to $\bigcup_{\phi \in \mathrm{Norm}(\mathscr{F})} \{\phi_*(V) \mid V \in \mathscr{F}\}$, where $\phi_*$ is the differential of $\phi$.

The second lemma gives the geometrical structure of the Lie saturate of a family of vector fields.

**Lemma 6.4.** If $\mathscr{F}$ is any family of smooth vector fields, then $\mathscr{F}$ is equivalent to the closed convex cone generated by $\{\lambda V \mid 0 \leqslant \lambda \leqslant 1, V \in \mathscr{F}\}$, where the closure is in the $C^\infty$ topology on compact sets of $\mathbf{R}^n$.

## 6.2. Control of the Finite Dimensional Approximations of Navier–Stokes

We are able now to prove the controllability property of our equations. We aim to prove that the control problem

$$\begin{cases} \dot{r}_{\mathbf{k}} - F_{r_{\mathbf{k}}}(r, s) = q^r_{\mathbf{k}} v^r_{\mathbf{k}} \\ \dot{s}_{\mathbf{k}} - F_{s_{\mathbf{k}}}(r, s) = q^s_{\mathbf{k}} v^s_{\mathbf{k}}, \end{cases} \tag{6.1}$$

where $F_{r_{\mathbf{k}}}$ and $F_{s_{\mathbf{k}}}$ are defined in (3.1) and (3.2), and the $3 \times 3$ matrices are defined in (3.1), is controllable, in the sense that for each initial state $(r_I, s_I) \in U$, for each final state $(r_F, s_F) \in U$ and for each time $T > 0$ there is a family of controls $(v^r_{\mathbf{k}}, v^s_{\mathbf{k}})_{\mathbf{k} \in \mathscr{N}}$, where $\mathscr{N}$ is the set of indices corresponding to the non-zero $q_{\mathbf{k}}$, such that the solution corresponding to that control starts at $t = 0$ in $(r_I, s_I)$ and arrives in $(r_F, s_F)$ at time $t = T$.

**Theorem 6.5.** Assume that the set $\mathscr{N}$ of non-zero components of the control is a determining set of indices, as defined in Section 5. Then system (6.1) is controllable in the sense given above.

*Proof.* First we show that $\mathfrak{U}_{\mathbf{k}} \subset \mathrm{LS}(\mathscr{F})$ for $\mathbf{k} \in \mathscr{N}$. Let $\lambda \in \mathbf{R}$ and $\mathbf{k} \in \mathscr{N}$ and take $Y_{\mathbf{k}} \in \mathfrak{U}_{\mathbf{k}}$, since

$$\lambda Y_{\mathbf{k}} = \lim_{n \to \infty} \frac{1}{n}(F_0 + n\lambda Y_{\mathbf{k}}),$$

it follows by Lemma 6.4 that $Y_{\mathbf{k}} \in \mathrm{LS}(\mathscr{F})$.

Now we aim to show the following claim: *if* $\mathbf{m}, \mathbf{n} \in \mathcal{K}_N$ *are linear independent indices with* $|\mathbf{m}| \neq |\mathbf{n}|$ *and* $\mathbf{m}+\mathbf{n} \in \mathcal{K}_N$, *and if* $\mathfrak{U}_{\mathbf{m}} \oplus \mathfrak{U}_{\mathbf{n}} \subset \mathrm{LS}(\mathcal{F})$, *then also* $\mathfrak{U}_{\mathbf{m}+\mathbf{n}} \subset \mathrm{LS}(\mathcal{F})$. If the claim is true, it follows that each $\mathfrak{U}_{\mathbf{k}}$ is contained in $\mathrm{LS}(\mathcal{F})$ and, since by the assumptions $\mathcal{N}$ is a determining set of indices, Corollary 6.2 applies and the proof is ended.

The proof of the claim now follows. From Lemma 4.2 we know that $\mathfrak{U}_{\mathbf{m}+\mathbf{n}}$ is spanned by $[[F_0, V], W]$, where $V \in \mathfrak{U}_{\mathbf{m}}$ and $W \in \mathfrak{U}_{\mathbf{n}}$. From property (iii) of Lemma 4.1, we have that

$$[[F_0, V], W] = \tfrac{1}{2}[[F_0, V+W], V+W]$$

and

$$-[[F_0, V], W] = \tfrac{1}{2}[[F_0, V-W], V-W].$$

Since by Lemma 6.4 $\mathrm{LS}(\mathcal{F})$ is convex, in order to prove the claim it is sufficient to show that $\lambda[[F_0, V], V]$ is in $\mathrm{LS}(\mathcal{F})$ for each $\lambda > 0$ and $V \in \mathfrak{U}_{\mathbf{m}} \oplus \mathfrak{U}_{\mathbf{n}}$.

So, let $\alpha \in \mathbf{R}$ and $V \in \mathfrak{U}_{\mathbf{m}} \oplus \mathfrak{U}_{\mathbf{n}}$, then $\phi(x) = \mathrm{e}^{\alpha V}(x)$ is in $\mathrm{Norm}(\mathcal{F})$ (see the proof of Theorem 2 of Jurdjevic and Kupka[10]), so that, by Lemma 6.3, $(\mathrm{e}^{\alpha V})_* (F_0) \in \mathrm{LS}(\mathcal{F})$. Now, since the coefficients of $F_0$ are polynomials of degree 2 and $V$ is a constant vector field, it follows that

$$(\mathrm{e}^{\alpha V})_* (F_0) = I + \alpha[V, F_0] + \frac{\alpha^2}{2}[V, [V, F_0]],$$

and so, for each $\lambda > 0$,

$$\lambda[V, [V, F_0]] = \lim_{\alpha \to \infty} \frac{\lambda}{\alpha^2} (\mathrm{e}^{\alpha V})_* (F_0) \in \mathrm{LS}(\mathcal{F}),$$

since $\mathrm{LS}(\mathcal{F})$ is closed. The theorem is proved. ▮

## 7. THE EXPONENTIAL CONVERGENCE

In this last section we prove Theorem 2.2 as a consequence of a general result by Meyn and Tweedie[13] (see Theorem 6.1). Before giving the statement of such theorem, we need to state some definitions. They will be given in a simplified form, adapted to our case, while the general statements can be found in the papers by Meyn and Tweedie.[12, 13]

A nonempty subset $C$ of the state space $U$ is a *petite* set for a Markov process with transition probabilities $P_t(\cdot,\cdot)$ if there are a non-trivial measure $\varphi$ and a probability distribution $a$ on $(0, \infty)$ such that

$$\int P_t(x,\cdot)\, a(dt) \geqslant \varphi \qquad \text{for all} \quad x \in C.$$

A function $V\colon U \to \mathbf{R}_+$ is a Lyapunov function for the process if $V(x) \to \infty$ as $|x| \to \infty$ and there are real constants $c > 0$ and $d$ such that

$$\mathscr{L}V(x) \leqslant -cV(x) + d$$

where $\mathscr{L}$ is the generator of the diffusion.

The kinetic energy

$$V(r, s) = \sum_{k \in \tilde{\mathscr{K}}} \sum_{i=1}^{3} (r_{\mathbf{k}}^{i^2} + s_{\mathbf{k}}^{i^2})$$

will play the role of the Lyapunov function in our case, as stated by the following lemma.

**Lemma 7.1.** For each $(r, s) \in U$,

$$\sum_{\mathbf{k} \in \tilde{\mathscr{K}}} \sum_{i=1}^{3} (r_{\mathbf{k}}^i E_{r_{\mathbf{k}}^i} + s_{\mathbf{k}}^i E_{s_{\mathbf{k}}^i}) = 0,$$

where the polynomial $E_{r_{\mathbf{k}}^i}$ and $E_{s_{\mathbf{k}}^i}$ are respectively the homogeneous part of degree 2 of the polynomials $F_{r_{\mathbf{k}}^i}$ and $F_{s_{\mathbf{k}}^i}$, defined in (3.1) and (3.2), and

$$\mathscr{L}V(r, s) \leqslant -2\nu V(r, s) + \sigma^2,$$

where $\mathscr{L}$ is the generator defined in (2.2) and $\sigma^2$ is the variance of the Brownian motion $B_t$.

*Proof.* The first property is an easy consequence of a property of the non-linear term of Navier–Stokes equations, namely $\int v \cdot (v \cdot \nabla)\, v = 0$. Indeed

$$\sum_{\mathbf{k} \in \tilde{\mathscr{K}}} (r_{\mathbf{k}} \cdot E_{r_{\mathbf{k}}} + s_{\mathbf{k}} \cdot E_{s_{\mathbf{k}}}) = \sum_{\mathbf{k} \in \tilde{\mathscr{K}}} \sum_{\substack{\mathbf{h}, \mathbf{l} \in \mathscr{K}_N \\ \mathbf{h}+\mathbf{l}=\mathbf{k}}} \operatorname{Im}[(\mathbf{k} \cdot u_{\mathbf{h}})(u_{\mathbf{l}} \cdot \overline{u_{\mathbf{k}}})] = \sum_{\mathbf{k} \in \mathscr{K}_N} u_{\mathbf{k}} \cdot E_{u_{\mathbf{k}}} \qquad (7.1)$$

where $u_{\mathbf{k}} = r_{\mathbf{k}} + \mathrm{i} s_{\mathbf{k}}$ and $E_{u_{\mathbf{k}}}$ is the non-linear part in Eq. (2.1), namely

$$E_{u_{\mathbf{k}}} = \sum_{\substack{\mathbf{h}, \mathbf{l} \in \mathscr{K}_N \\ \mathbf{h}+\mathbf{l}=\mathbf{k}}} (\mathbf{k} \cdot u_{\mathbf{h}}) \left( u_{\mathbf{l}} - \frac{\mathbf{k} \cdot u_{\mathbf{l}}}{|\mathbf{k}|^2}\, \mathbf{k} \right).$$

Finally, the proof that the last sum in (7.1) is equal to 0 is just a matter of swapping the two indices **k** and **l**.

The second property is then an easy consequence of the previous one:

$$\mathscr{L}V = \sum_{\mathbf{k} \in \tilde{\mathscr{K}}} (-2\nu\,|\mathbf{k}|^2\,(r_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) + r_{\mathbf{k}} \cdot E_{r_{\mathbf{k}}} + s_{\mathbf{k}} \cdot E_{s_{\mathbf{k}}}) + \sigma^2 \leqslant -2\nu V + \sigma^2. \quad \blacksquare$$

Now we are able to prove Theorem 2.2.

*Proof of Theorem 2.2.* From Theorem 2.1 we know that the Markov process $(r(t), s(t))$ is strong Feller and irreducible. Using Theorems 3.3 and 4.1 of Meyn and Tweedie,[12] it follows that all compact sets of the state space $U$ are petite sets. Moreover the previous lemma tells us that the kinetic energy $V$ is a Lyapunov function. By means of Theorem 6.1 of Meyn and Tweedie,[13] we conclude that there are positive constants $C$ and $\rho$ such that for each $(r_0, s_0) \in U$,

$$\|P_t((r_0, s_0), \cdot) - \pi\|_f \leqslant C\left(1 + V(r_0, s_0) + \frac{\sigma^2}{2\nu}\right) e^{-\rho t}$$

with $f = 1 + V$. $\blacksquare$

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